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On the termination of one-dimensional transport in transient buoyancy induced flow adjacent to a vertical surface

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1. INTRODUCTION

TRANSIENT natural convection resulting from a sudden change in the surface condition has been studied by various investigators. Many of these studies were recently summarized [1]. In all such transients, the initial temperature $t(y,\tau)$ and velocity $u(y,\tau)$ are purely diffusive. Convective effects propagate downstream of the leading edge at a finite rate, resulting eventually in steady boundary region transport.

Goldstein and Briggs [2] proposed a model for the termination of one-dimensional transport based on a maximum leading edge penetration distance, x_{GB}, at a given time, τ, where

$$x_{GB}(\tau) = \max \int_0^\tau u(y,\xi)\,d\xi. \quad (1)$$

Yang [3] and later Nanbu [4] examined the boundary layer form of the governing transient equations. They concluded that the departure from the purely diffusive transport occurs with the appearance of an essential singularity in the equations at a critical time. Brown and Riley [5] pointed out that this critical time yields a different leading edge propagation criterion than equation (1)

$$x_{BR}(\tau) = \int_0^\tau \max\,(u(y,\xi))\,d\xi. \quad (2)$$

Calculations for the sudden change in surface temperature condition for $Pr = 1$ yielded only slightly faster propagation rates than equation (1).

Many experimental investigations have attempted to verify the above propagation criteria in gases [6–8] and liquids [1]. However, the measured propagation distances have been significantly larger than those predicted by the models.

None of the studies listed above has analyzed in detail the implications of using equations (1) and (2) on the transient development of various transport quantities. In the present study, four different propagation criteria are obtained by first assuming and later justifying no overshoots from eventual steady values in selected transport quantities. The physical quantities are surface temperature, mass flow rate, maximum tangential velocity and surface shear stress.

Propagation rates have been computed for a sudden input of internal energy generation rate within a vertical surface. At the surface

$$c''\frac{\partial t}{\partial \tau}(0,\tau) - k\frac{\partial t}{\partial y}(0,\tau) = q''_\infty \quad \text{for} \quad \tau > 0. \quad (3)$$

The calculations presented here clearly show that propagation rates must be significantly faster than those predicted by either equation (1) or equation (2) to avoid unrealistic overshoots in transport quantities. The different analytical propagation rate models are also compared with all available experimental data.

The mechanisms of departure from one-dimensional transport considered here are purely buoyancy induced in nature. Disturbances having a thermoacoustic origin are not considered, since the resulting frequencies and arrival times are several orders of magnitude faster than actual measured times. In a numerical study of transient natural convection in enclosures at varying gravity levels, Spradley and Churchill [9] have found these thermoacoustic effects on transport to be small at terrestrial intensity of gravity.

2. ADDITIONAL PROPAGATION CRITERIA

The following criteria are obtained by first postulating that the physical quantity being considered does not overshoot its corresponding steady level, during the one-dimensional transport. It will be shown in Section 3 that these criteria automatically hold when unrealistic overshoots in the mass flow rate are not permitted. In the following, one-dimensional transport quantities have been obtained for the surface condition in equation (3) from Table 1 of Goldstein and Briggs [2] and the steady, self-similar solutions for the uniform flux surface from Gebhart [10].

2.1. *Propagation distances for no overshoot in surface temperature*

This requirement implies at any downstream x

$$(t_{0_{ss}}(x,0) - t_\infty) \geqslant (t_0(0,\tau) - t_\infty). \quad (4)$$

Equation (4) can be rearranged using expressions for $t_0(0,\tau)$ and $t_{0_{ss}}$. After some simplification

$$x_1 \geqslant \left(\frac{1}{4}\right)\frac{(-\phi'(0))^4}{Pr^2}\frac{g\beta q''_\infty}{k}(\alpha\tau)^{1/2}\tau^2(F_1(a\tau^{1/2}))^5 \quad (5)$$

where

$$F_1(a\tau^{1/2}) = \frac{2}{\sqrt{\pi}} + \frac{e^{a^2\tau}\,\text{erfc}\,a\tau^{1/2}}{a\tau^{1/2}} - \frac{1}{a\tau^{1/2}}. \quad (6)$$

In equation (5)

$$\phi'(0) = \left\{\frac{\partial t}{\partial y}(x,0)\right\} \Big/ \{(t_0 - t_\infty)b(x)\}.$$

2.2. *Propagation distances for no overshoot in mass flow rate*

The required condition is

$$\int_0^\infty u_{ss}(x,y)\,dy \geqslant \int_0^\infty u(y,\tau)\,dy. \quad (7)$$

This results in

$$x_2 \geqslant \left(\frac{g\beta q''_\infty}{k}\right)(\alpha\tau)^{1/2}\tau^2\left[\frac{1}{2Pr}\right]^{3/4}\left[\frac{\{-\phi'(0)\}^{1/5}F_2(a\tau^{1/2})}{(1+\sqrt{Pr})f(\infty)}\right]^{5/4} \quad (8)$$

NOMENCLATURE

a — $(\rho c_p k)^{1/2}/c''$
b — $(Gr_x/4)^{1/4}/x$
c'' — thermal capacity of the surface, per unit area exposed to fluid
c_p — specific heat of fluid
g — acceleration due to gravity
Gr_x — Grashof number, $g\beta x^3(t_0-t_\infty)/v^2$
$\mathrm{i}^n\mathrm{erfc}\,z$ — nth integral of complimentary error function
k — thermal conductivity of fluid
Pr — Prandtl number
q''_∞ — steady surface heat flux
t — temperature
u — component of velocity parallel to the surface
x — downstream distance
x_{BR} — propagation distance from equation (2)
x_{GB} — propagation distance from equation (1)
x_i — propagation distance for model i; $i = 1$–4
x'_i — non-dimensional propagation distance
y — normal distance from surface
y_{max} — normal location of velocity maximum.

Greek symbols
α — thermal diffusivity
β — coefficient of thermal expansion
η — non-dimensional horizontal coordinate, $y/x(Gr_x/4)^{1/4}$
μ — dynamic viscosity
v — kinematic viscosity
ξ — non-dimensional location, $y_{max}/2\sqrt{(\alpha\tau)}$
ρ — density
τ — time
ψ — stream function.

Subscripts
0 — at the surface
∞ — in the ambient fluid
ss — in steady state.

where

$$F_2(a\tau^{1/2}) = \frac{1}{4} - \frac{2}{3\sqrt{\pi}a\tau^{1/2}} + \frac{1}{2a^2\tau} - \frac{1}{\sqrt{\pi}a^3\tau^{3/2}} + \frac{(1-\mathrm{e}^{a^2\tau}\,\mathrm{erfc}\,a\tau^{1/2})}{2a^4\tau^2}. \quad (9)$$

In equation (8), $f(\infty)$ is the value of the non-dimensional stream function $f(\eta) = \psi(x,y)/4v(Gr_x/4)^{1/4}$, as $\eta \to \infty$.

2.3. *Propagation distances for no overshoot in maximum velocity*

This requires that the maximum tangential velocity from the short time solution not exceed the eventual maximum steady velocity at a given downstream location. Mathematically

$$\max(u_{ss}(x,y)) \geqslant \max(u(y,\tau)). \quad (10)$$

In equation (10), both maximizations are over y. The above condition implies

$$x_3 \geqslant \frac{g\beta q''_\infty(\alpha\tau)^{1/2}\tau^2}{4k\,Pr^{1/3}}\left[\frac{F_{3,1}(\xi, Pr, a\tau^{1/2})}{(1-Pr)f'_{max}}\right]^{5/3}(-\phi'(0))^{2/3} \quad (\text{for } Pr \neq 1)$$

$$\geqslant \frac{g\beta q''_\infty(\alpha\tau)^{1/2}\tau^2}{4k}\left[\frac{F_{3,2}(\xi, a\tau^{1/2})}{f'_{max}}\right]^{5/3}(-\phi'(0))^{2/3} \quad (\text{for } Pr = 1) \quad (11a,b)$$

where

$$F_{3,1}(\xi, Pr, a\tau^{1/2}) = \Bigg(8(\mathrm{i}^3\,\mathrm{erfc}\,\xi - \mathrm{i}^3\,\mathrm{erfc}\,\xi/\sqrt{Pr}) + \frac{1}{a^3\tau^{3/2}}[\mathrm{e}^{a^2\tau+2a\tau^{1/2}\xi}\,\mathrm{erfc}\,(\xi+a\tau^{1/2}) - \mathrm{e}^{a^2\tau+2a\tau^{1/2}\xi/\sqrt{Pr}}\,\mathrm{erfc}\,(\xi/\sqrt{Pr}+a\tau^{1/2})] - \frac{1}{a^3\tau^{3/2}}\left\{\sum_{r=0}^{2}(-2a\tau^{1/2})^r\mathrm{i}^r\,\mathrm{erfc}\,\xi - \sum_{r=0}^{2}(-2a\tau^{1/2})^r\mathrm{i}^r\,\mathrm{erfc}\,\xi/\sqrt{Pr}\right\}\Bigg) \quad (12)$$

and

$$F_{3,2}(\xi, a\tau^{1/2}) = (4\xi\,\mathrm{i}^2\mathrm{erfc}\,\xi + \frac{\xi}{a^2\tau}(\mathrm{erfc}\,\xi - 2a\tau^{1/2}\,\mathrm{ierfc}\,\xi) - \frac{\xi}{a^2\tau}\mathrm{e}^{a^2\tau+2a\tau^{1/2}\xi}\,\mathrm{erfc}\,(\xi+a\tau^{1/2})). \quad (13)$$

In equations (11a) and (11b), f'_{max} is the maximum non-dimensional velocity of the steady state boundary region flow given by (see ref. [10])

$$f'_{max} = \frac{\max(u_{ss}(x,y))}{(4v/x)(Gr_x/4)^{1/2}}. \quad (14)$$

Also, ξ is the non-dimensional location of the maximum one-dimensional velocity from Goldstein and Briggs [2].

2.4. *Propagation distances for no overshoot in surface shear stress*

The required condition is

$$\mu\frac{\partial u}{\partial y}(x,0) \geqslant \mu\frac{\partial u}{\partial y}(0,\tau). \quad (15)$$

This can be recast as

$$x_4 \geqslant \frac{g\beta q''_\infty}{k}(\alpha\tau)^{1/2}\tau^2\frac{(-\phi'(0))^{3/2}}{4Pr^{3/4}}\left[\frac{F_4(a\tau^{1/2})}{(1+\sqrt{Pr})f''(0)}\right]^{5/2} \quad (16)$$

where

$$F_4(a\tau^{1/2}) = 1 - \frac{2}{a\tau^{1/2}\sqrt{\pi}} - \frac{1}{a^2\tau}(\mathrm{e}^{a^2\tau}\,\mathrm{erfc}\,a\tau^{1/2} - 1). \quad (17)$$

In equation (16)

$$f''(0) = \left(\frac{\partial u_{ss}}{\partial y}(x,0)\right)x^2\Big/(4v(Gr_x/4)^{3/4}).$$

2.5. *Asymptotic behavior of very small and very large thermal capacity surfaces*

For $a\tau^{1/2} \gg 1$ and $a\tau^{1/2} \ll 1$, the functions F_1, F_2, $F_{3,1}$, $F_{3,2}$ and F_4 can be simplified. For $a\tau^{1/2} < 1$, the following series is utilized

$$\mathrm{e}^{a^2\tau+2a\tau^{1/2}\xi}\,\mathrm{erfc}\,(\xi+a\tau^{1/2}) = \mathrm{erfc}\,\xi - 2a\tau^{1/2}\,\mathrm{ierfc}\,\xi + 4a^2\tau\,\mathrm{i}^2\mathrm{erfc}\,\xi - \cdots \quad (18)$$

Also, as $a\tau^{1/2} \to \infty$

$$e^{a^2\tau}\,\mathrm{erfc}\,a\tau^{1/2} \to 0. \tag{19}$$

3. COMPUTATIONS AND RESULTS

Lower bounds on x_1–x_4 were computed using the formulation in Section 2. These distances were then non-dimensionalized in the following manner

$$x_i' = x_i k/g\beta q_x''(\alpha\tau)^{1/2}\tau^2; \quad \text{for} \quad i = 1\text{–}4. \tag{20}$$

Both x_1'–x_4' and equation (1) are plotted in Figs. 1 and 2 for $Pr = 0.72$, nitrogen, and $Pr = 6.7$, water. Equation (2) is only about 5% higher and is not shown to preserve clarity. These figures provide estimates of propagation distances for any surface of known thermal capacity. Asymptotes for $a\tau^{1/2} \to \infty$, surfaces with negligible thermal capacity, are shown as short horizontal lines in the upper right region. Also shown in Figs. 1 and 2 are samples of propagation rate measurements from all previous studies.

In gases, Fig. 1 shows that for $a\tau^{1/2} \leqslant 8$, equation (1) gives the slowest propagation rate of all models. The fastest rate is given by the requirement in Section 3.2 that no overshoot occur in the mass flow rate. This also results in no overshoot in any of the other three transport quantities considered. Looked at in another way, equation (1) always predicts an overshoot in the mass flow rate. For a Boussinesq fluid, such overshoot is unrealistic. An efflux of mass from the flow region must take place to remove the excess entrained fluid. This has not been observed in any experimental studies of laminar transients.

It is clear from Fig. 1 that for $a\tau^{1/2} \leqslant 8$, x_1' yields the slowest propagation rate. This implies that no overshoot in the surface temperature would be predicted during the one-dimensional transport, from any of the other criteria. This is in agreement with all experimental studies to date.

The experimental data [6–8] in Fig. 1 are in the range of $a\tau^{1/2} = 0.4$–1.4. In this range, all propagation criteria show a strong dependence on $a\tau^{1/2}$. Depending upon $a\tau^{1/2}$, the propagation distance x_2' is 1.26–1.53 times x_{GB}'. All experimental points are above x_2', those from Mahajan and Gebhart [8] being significantly higher. The considerable scatter in the data is due, in part, to the inability to sharply identify the first departures from the short-time transport [6]. It is also noted that x_2', provides only the lower limit on the propagation rate. If the one-dimensional phase is only a small fraction of the total transient time, as is true of measurements in Fig. 1, the conductive transport will end before the mass flow rate reaches the steady value.

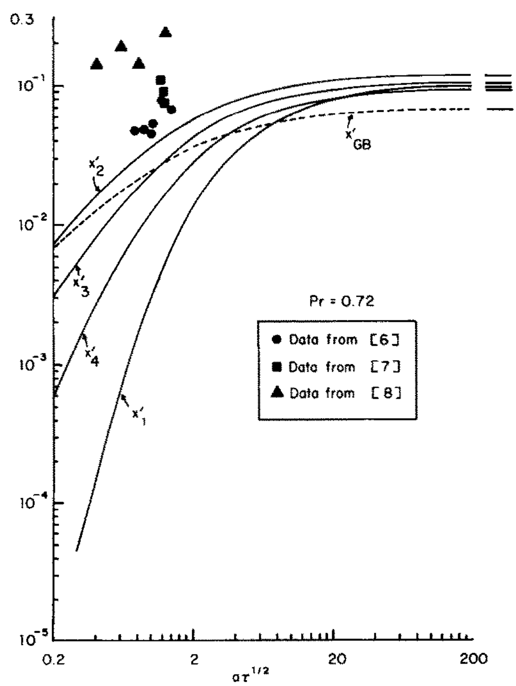

FIG. 1. Non-dimensional propagation distances in air.

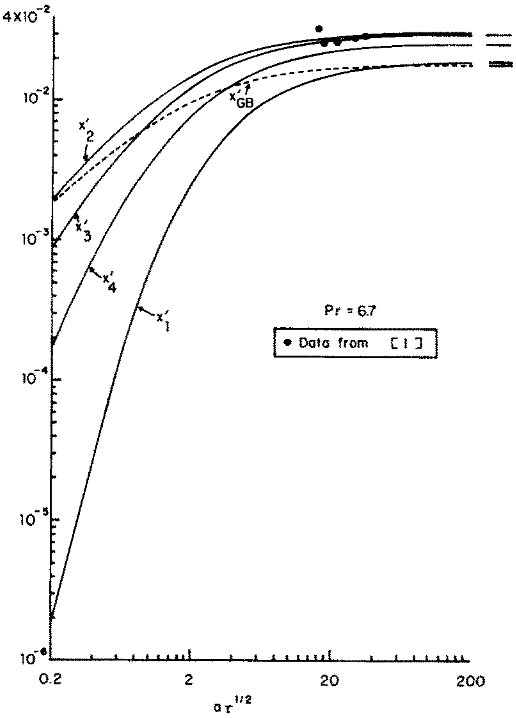

FIG. 2. Non-dimensional propagation distances in water.

The data from refs. [6, 7] are interferometer measurements of first departure times for surface heat transfer. Those from ref. [8] are sensor measurements at the outer edge of the transport region. As is clear from Fig. 1, when computation of surface transport is of primary interest, equation (8) can be used with reasonable accuracy.

Propagation distances for water, $Pr = 6.7$, are shown in Fig. 2. As for gases, the criterion requiring no overshoot in mass flow rate gives the fastest propagation. Use of equation (1) again gives unrealistic overshoots in the mass flow rate. Also shown in Fig. 2 are data from ref. [1], which fall in the range of $a\tau^{1/2} = 16$–34. In contrast to Fig. 1, in this range, all models show only a weak dependence on $a\tau^{1/2}$. This is to be expected, due to the much smaller surface thermal capacity effects in water [1].

The data in Fig. 2 have been obtained both from flow visualizations and local hot film and thermocouple measurements. Excellent agreement is seen between the data and the model from Section 2.2. The transient times ranged from 20 to 90 s, with the one-dimensional transport occupying a very large fraction of the entire transient. A much sharper determination of the first departure times was therefore possible.

4. CONCLUSIONS

Based on the computations presented above, it is clear that the propagation models based on either equation (1) or equation (2) result in unrealistic overshoots in the mass flow rate. The faster propagation rates predicted by the model in Section 2.2 show good agreement with data in water. In nitrogen, the data are higher even though they exhibit significant scatter. For both liquids and gases, however, equation (8) provides a better estimate of the one-dimensional transport termination times than any other previous criteria.

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A note on the solution of the free convection boundary layer flow in a saturated porous medium

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INTRODUCTION

BUOYANCY driven flow past bodies immersed in a saturated porous medium has been the subject of the studies by Cheng [1–3]. The effect of uniform mass flux on the free convection boundary layer on a vertical wall in a saturated porous medium was studied by Merkin [4]. Cheng [5] presented a similarity solution for the case of wall temperature and suction velocity varying as powers of x, the longitudinal distance. In all these problems numerical solutions have been given for selected values of a parameter. In this note, an analytical series solution based on Von Mises transformation is given for the problem studied by Cheng [5]. It is found that even a few terms of the series are sufficient to yield the numerical results reported in ref. [5]. The treatment is similar to that of Merkin [6]. However, the equation and the boundary conditions are different.

THE EQUATION AND SERIES SOLUTION

The boundary layer equations of momentum and energy for the boundary layer flow past a vertical plate embedded in a saturated porous medium can be reduced to the form [5]

$$f''-\theta' = 0 \tag{1}$$

$$\theta''+\frac{1+\lambda}{2}f\theta'-\lambda f'\theta = 0 \tag{2}$$

where the plate temperature and suction or injection velocity are respectively given by

$$T_w - T_\infty = Ax^\lambda \tag{3}$$

and

$$v_w = ax^n \tag{4}$$

where

$$n = (\lambda-1)/2.$$

The boundary conditions are

$$\begin{aligned} &\eta = 0: \quad \theta = 1, \quad f = f_w \\ &\eta \to \infty: \theta = 0, \quad f' = 0 \end{aligned} \tag{5}$$

where f_w is the non-dimensional form of suction ($f_w > 0$) or injection ($f_w < 0$) velocity. Eliminating θ gives

$$f'''+\frac{1+\lambda}{2}ff''-\lambda(f')^2 = 0. \tag{6}$$

Following Merkin [6], we transform equation (6), expressing $p = p(\phi)$ where

$$p = f'(\eta), \quad \phi = c-f(\eta), \quad c = f(\infty). \tag{7a–c}$$

The modified equation is

$$\frac{d}{d\phi}\left(p\frac{dp}{d\phi}\right)+\frac{1+\lambda}{2}(\phi-c)\frac{dp}{d\phi}-p = 0. \tag{8}$$

The boundary conditions on p are

$$\begin{aligned} &\phi = 0: \quad & p = 0 \\ &\phi = c-f_w: \quad & p = 1. \end{aligned} \tag{9}$$

We expand p in the series form

$$p = \sum_i a_i\phi^i. \tag{10}$$